MIDTERM: ALGEBRA I

Date: **12th September 2017**
The Total points is **108** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (28 points) State true or false. No justification needed.
  (a) Let R_1 and R_2 be integral domains. Then $R_1 \times R_2$ is an integral domain.
  (b) Let R_1 and R_2 be reduced rings. Then $R_1 \times R_2$ is a reduced ring.
  (c) A Unique Factorization Domain is a Principal Ideal Domain.
  (d) A prime element in a ring is irreducible.
  (e) Every module over the ring $\mathbb{Z}/5\mathbb{Z}$ is torsion free.
  (f) Let M be a finitely generated R-module then $S^{-1}M$ is a finitely generated R-module.
  (g) Every short exact sequence over the ring $\mathbb{Z}/4\mathbb{Z}$ splits.

(2) (13 + 7=20 points) Let R be a ring and S a multiplicative subset. Show that prime ideals of $S^{-1}R$ are in one to one bijection with prime ideals of R disjoint with S. Does this statement hold with prime replaced by maximal? Justify your answer.

(3) (5+7+8=20 points) State Eisenstein's criterion for irreducibility. Show that the polynomial $f(X,Y,Z) = X^2Y^2Z^2 + Z^4 - Y^3Z + X^2Y$ is an irreducible polynomial in $R = \mathbb{C}[X,Y,Z]$. Let I be the ideal $(U - f(X,Y,Z))$ in $R[U]$. Show that $R[U]/I$ is a UFD.

(4) (5+15=20 points) Define faithful module. Let $R[x]$ be a polynomial ring over a ring R and M be a faithful $R[x]$-module. Show that if M is a finitely generated R-module then $M = 0$.

(5) (5+15=20 points) Let R be a ring. What is a short exact sequence of R-modules? Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Let F be a finitely generated free R-module. Show that the following induced sequence is exact.

$$0 \to \mathrm{Hom}_R(F,A) \to \mathrm{Hom}_R(F,B) \to \mathrm{Hom}_R(F,C) \to 0$$